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REMARKS ON THE PAPER: ORTHOGONALLY ADDITIVE AND ORTHOGONALLY QUADRATIC FUNCTIONAL EQUATION

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Abstract. The main goal of this paper is to present the additional stability results of the following orthogonally additive and orthogonally quadratic functional equation

$$\begin{aligned} &f(\frac{x}{2}+y)+f(\frac{x}{2}-y)+f(\frac{x}{2}+z)+f(\frac{x}{2}-z) \\ &= \frac{3}{2}f(x)-\frac{1}{2}f(-x)+f(y)+f(-y)+f(z)+f(-z), \end{aligned}$$

for all x, y, z with $x \perp y$, which has been introduced in the paper [11], in orthogonality Banach spaces and in non-Archimedean orthogonality Banach spaces.

1. Introduction

In 1897, K. Hensel [7] has provided a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications([2, 9, 10, 15]). Let $\mathbb{K}$ be a field equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0,\infty)$. The field $\mathbb{K}$ is called a *non-Archimedean field* if the function $|\cdot| : \mathbb{K} \to [0,\infty)$, called the non-Archimedean valuation, satisfies the following conditions:

(1) $|r| = 0$ if and only if $r = 0$;
(2) $|rs| = |r||s|$ $(r \in \mathbb{K}, x \in X)$;
(3) the strong triangle inequality: $|r + s| \le \max\{|r|, |s|\}$, $\forall r, s \in \mathbb{K}$.

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Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

Let X be a vector space over a non-Archimedean field $\mathbb{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0, \infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

(1) $\|x\| = 0$ if and only if $x = 0$;
(2) $\|rx\| = |r|\|x\| \quad (r \in \mathbb{K},\, x \in X)$;
(3) the strong triangle inequality

$$\|x + y\| \leq \max\{\|x\|, \|y\|\},\ \forall x, y \in X.$$

In this case, $(X, \|\cdot\|)$ is called a *non-Archimedean normed space.*

Definition 1.1. Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer n_0 such that

$$\|x_n - x_m\| \leq \varepsilon$$

for all $n, m \geq n_0$. The sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there is a positive integer n_0 and an $x \in X$ such that

$$\|x_n - x\| \leq \varepsilon$$

for all $n \geq n_0$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$. If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space.*

A.G. Pinsker [16] have investigated properties of orthogonally additive functionals on inner product spaces. K. Sundaresan [20] generalized these results to arbitrary Banach spaces equipped with the Birkhoff-James orthogonality [1, 8]. The orthogonal Cauchy functional equation

$$f(x + y) = f(x) + f(y), \quad x \perp y,$$

was first investigated by S. Gudder and D. Strawther [6], where $\perp$ is an abstract orthogonality relation.

In 1985, J. Rätz [18] introduced a new definition of orthogonality by using more restrictive axioms than those of S. Gudder and D. Strawther. Moreover, he investigated the structure of orthogonally additive mappings. J. Rätz and Gy. Szabó [19] investigated the problem in a rather more general framework.

We introduce the definition of the orthogonality space in the sense of J. Rätz; cf. [18].

Suppose X is a real vector space with $\dim X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

(O_1) totality of $\perp$ for zero: $x \perp 0, 0 \perp x$ for all $x \in X$;
(O_2) independence: if $x, y \in X - 0, x \perp y$, then x, y are linearly independent;
(O_3) homogeneity: if $x, y \in X - 0, x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;
(O_4) the Thalesian property: if P is a 2-dimensional subspace of X, $x \in P$ and $\lambda \in \mathbb{R}_+$, which is the set of nonnegative real numbers, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

In this case, the pair $(X, \perp)$ is called an orthogonality space and we denote an orthogonality normed space by an orthogonality space with a norm. There are some well-known interesting examples as follows:

(i) The trivial orthogonality on a vector space X defined by (O_1), and for non-zero elements $x, y \in X, x \perp y$ if and only if x, y are linearly independent.
(ii) The ordinary orthogonality on an inner product space $(X, \langle , \rangle)$ given by $x \perp y$ if and only if $< x, y >= 0$.
(iii) The Birkhoff-James orthogonality on a normed space $(X, ||.||)$ defined by $x \perp y$ if and only if $||x + \lambda y|| \geq ||x||$ for all $\lambda \in \mathbb{R}$.

R. Ger and J. Sikorska [5] have proved the orthogonal stability of the Cauchy functional equation $f(x+y) = f(x)+f(y)$, namely, they proved that if f is a mapping from an orthogonality space X into a real Banach space Y and $||f(x+y)-f(x)-f(y)|| \leq \varepsilon$ for all $x, y \in X$ with $x \perp y$ and some $\varepsilon > 0$, then there exists exactly one orthogonally additive mapping $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{16}{3}\varepsilon$ for all $x \in X$.

Now, the following orthogonally quadratic equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \quad x \perp y$$

was first investigated by F. Vajzović [22] when X is a Hilbert space, Y is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. After that H. Drljević [3], M. Fochi [4], M.S. Moslehian [12, 13], M.S. Moslehian and Th.M. Rassias [14], L. Paganoni and J. Rätz [17] and Gy. Szabó [21] generalized this result.

The authors in the paper [11] have proved the stability results for the following orthogonally additive and orthogonally quadratic functional equation

$$\begin{aligned} 0 &= Df(x, y, z) \\ &:= f(\frac{x}{2} + y) + f(\frac{x}{2} - y) + f(\frac{x}{2} + z) + f(\frac{x}{2} - z) \\ &\quad -\frac{3}{2}f(x) + \frac{1}{2}f(-x) - f(y) - f(-y) - f(z) - f(-z) \end{aligned}$$

for all $x, y, z \in X$ with $x \perp y$. If a mapping f with $Df(x, y, z) = 0$ is odd, then

$$f(\frac{x}{2}+y)+f(\frac{x}{2}-y)+f(\frac{x}{2}+z)+f(\frac{x}{2}-z)=2f(x),$$

and if f is an even mapping satisfying $Df(x, y, z) = 0$, then

$$f(\frac{x}{2}+y)+f(\frac{x}{2}-y)+f(\frac{x}{2}+z)+f(\frac{x}{2}-z)=f(x)+2f(y)+2f(z),$$

for all $x, y, z \in X$ with $x \perp y$. Therefore the authors [11] have introduced the following definitions.

Definition 1.2. [11] A mapping $f : X \to Y$ is called an *orthogonally additive mapping* if

$$f(\frac{x}{2}+y)+f(\frac{x}{2}-y)+f(\frac{x}{2}+z)+f(\frac{x}{2}-z)=2f(x)$$

for all $x, y, z \in X$ with $x \perp y$.

Definition 1.3. [11] A mapping $f : X \to Y$ is called an *orthogonally quadratic mapping* if

$$f(\frac{x}{2}+y)+f(\frac{x}{2}-y)+f(\frac{x}{2}+z)+f(\frac{x}{2}-z)=f(x)+2f(y)+2f(z)$$

for all $x, y, z \in X$ with $x \perp y$.

In this paper, we are going to introduce some additional stability results of the orthogonally additive and orthogonally quadratic functional equation $Df(x, y, z) = 0$.

2. Approximate orthogonally additive and orthogonally quadratic mappings

Throughout this section, assume that $(X, \perp)$ is an orthogonality space and that $(Y, \|\cdot\|_Y)$ is a Banach space.

We observe that if a mapping f satisfies

$$f(\frac{x}{2}+y)+f(\frac{x}{2}-y)+f(\frac{x}{2}+z)+f(\frac{x}{2}-z)=2f(x)$$

for all $x, y, z \in X$ with $x \perp y$, then we easily see that (i) $f(0) = 0$; (ii) $f(-y) = -f(y)$; (iii) $f(\frac{x}{2}) = \frac{1}{2}f(x)$; (iv) $f(\frac{x}{2}+z)+f(\frac{x}{2}-z) = f(x)$ for all $x, y, z \in X$, and so f is additive. The converse is trivially true. Similarly, if a mapping $f : X \to Y$ satisfies

$$f(\frac{x}{2}+y)+f(\frac{x}{2}-y)+f(\frac{x}{2}+z)+f(\frac{x}{2}-z)=f(x)+2f(y)+2f(z)$$

for all $x, y, z \in X$ with $x \perp y$, then we obtain that (i) $f(0) = 0$; (ii) $f(-y) = f(y)$; (iii) $f(\frac{x}{2}) = \frac{1}{4}f(x)$; (iv) $f(\frac{x}{2}+z)+f(\frac{x}{2}-z) = \frac{x}{2}f(x)+2f(z)$ for all $x, y, z \in X$, and so f is quadratic. The converse is trivially true.

At first, we state some stability results of the functional equation $Df(x, y, z) = 0$ in the reference [11].

Theorem 2.1. [11] *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $0 < \alpha_1 < 1$ ($0 < \alpha_2 < 1$, resp.) with*

$$\begin{aligned} \varphi(x, y, z) &\leq 2\alpha_1\varphi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}), \\ \big(\varphi(x, y, z) &\leq \frac{\alpha_2}{2}\varphi(2x, 2y, 2z),\ \textit{resp.}\big) \end{aligned}$$

for all $x, y, z \in X$ with $x \perp y$. Let $f : X \to Y$ be an odd mapping satisfying

$$\|Df(x, y, z)\|_Y \leq \varphi(x, y, z) \tag{2.1}$$

for all $x, y, z \in X$ with $x \perp y$. Then there exists a unique orthogonally additive mapping $L_1 : X \to Y$ ($L_2 : X \to Y$, resp.) such that

$$\begin{aligned} \|f(x) - L_1(x)\|_Y &\leq \frac{\alpha_1}{2 - 2\alpha_1}\varphi(x, 0, 0), \\ \big(\|f(x) - L_2(x)\|_Y &\leq \frac{1}{2 - 2\alpha_2}\varphi(x, 0, 0),\ \textit{resp.}\big) \end{aligned}$$

for all $x \in X$.

Theorem 2.2. [11] *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $0 < \alpha_3 < 1$ ($0 < \alpha_4 < 1$, resp.) with*

$$\begin{aligned} \varphi(x, y, z) &\leq 4\alpha_3\varphi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}), \\ \big(\varphi(x, y, z) &\leq \frac{\alpha_4}{4}\varphi(2x, 2y, 2z),\ \textit{resp.}\big) \end{aligned}$$

for all $x, y, z \in X$ with $x \perp y$. Let $f : X \to Y$ be an even mapping satisfying (2.1). Then there exists a unique orthogonally quadratic mapping $Q_1 : X \to Y$ ($Q_2 : X \to Y$, resp.) such that

$$\begin{aligned} \|f(x) - Q_1(x)\|_Y &\leq \frac{\alpha_3}{1 - \alpha_3}\varphi(x, 0, 0), \\ \big(\|f(x) - Q_2(x)\|_Y &\leq \frac{1}{1 - \alpha_4}\varphi(x, 0, 0),\ \textit{resp.}\big) \end{aligned}$$

for all $x \in X$.

THEOREM 2.3. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $0 < \alpha < 1$ with*

$$\varphi(x, y, z) \leq 2\alpha\varphi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2})$$

for all $x, y, z \in X$ with $x \perp y$. Let $f : X \to Y$ be a mapping satisfying

$$\|Df(x, y, z)\|_Y \leq \varphi(x, y, z) \tag{2.2}$$

for all $x, y, z \in X$ with $x \perp y$. Then there exist an orthogonally additive mapping $L_1 : X \to Y$ and an orthogonally quadratic mapping $Q_1 : X \to Y$ such that

$$\begin{aligned} &\|f(x) - L_1(x) - Q_1(x)\|_Y \\ &\leq \Big(\frac{\alpha}{4(1-\alpha)} + \frac{\alpha}{2(2-\alpha)}\Big)[\varphi(x, 0, 0) + \varphi(-x, 0, 0)] \end{aligned} \tag{2.3}$$

for all $x \in X$. The functions L_1 and Q_1 are given by

$$L_1(x) = \lim_{k\to\infty} \frac{1}{2^k} f(2^k x), \quad Q_1(x) = \lim_{k\to\infty} \frac{1}{2^{2k}} f(2^k x),$$

for all $x \in X$.

Proof. Let $f_o(x) = \frac{f(x)-f(-x)}{2}$ and $f_e(x) = \frac{f(x)+f(-x)}{2}$. Then f_o is an odd mapping and f_e is an even mapping such that $f = f_o + f_e$. From (2.2), we get that

$$\begin{aligned} \|Df_o(x, y, z)\|_Y &\leq \frac{1}{2}[\varphi(x, y, z) + \varphi(-x, -y, -z)], \\ \|Df_e(x, y, z)\|_Y &\leq \frac{1}{2}[\varphi(x, y, z) + \varphi(-x, -y, -z)] \end{aligned} \tag{2.4}$$

for all $x, y, z \in X$ with $x \perp y$.

Then by Theorem 2.1 ($\alpha_1 := \alpha$) and by Theorem 2.2 ($\alpha_3 := \frac{\alpha}{2}$), there exist a unique orthogonally additive mapping $L_1 : X \to Y$, defined by $L_1(x) = \lim_{k\to\infty} \frac{1}{2^k} f(2^k x)$, and a unique orthogonally quadratic mapping $Q_1 : X \to Y$, defined by $Q_1(x) = \lim_{k\to\infty} \frac{1}{2^{2k}} f(2^k x)$, such that

$$\begin{aligned} \|f_o(x) - L_1(x)\|_Y &\leq \frac{\alpha}{(1-\alpha)4}[\varphi(x, 0, 0) + \varphi(-x, 0, 0)], \\ \|f_e(x) - Q_1(x)\|_Y &\leq \frac{\alpha}{(2-\alpha)2}[\varphi(x, 0, 0) + \varphi(-x, 0, 0)] \end{aligned}$$

for all $x \in X$, respectively. Therefore, we obtain the desired inequality (2.3), which completes the proof. $\square$

Corollary 2.4. [11] *Assume that* $(X, \perp)$ *is an orthogonality normed space. Let* θ *be a positive real number and* p *a real number with* $0 < p < 1$. *Let* $f : X \to Y$ *be a mapping satisfying*

$$\|Df(x,y,z)\|_Y \le \theta(\|x\|^p + \|y\|^p + \|z\|^p) \tag{2.5}$$

for all $x, y, z \in X$ *with* $x \perp y$. *Then there exist orthogonally additive mapping* $L_1 : X \to Y$ *and orthogonally quadratic mapping* $Q_1 : X \to Y$ *such that*

$$\|f(x) - L_1(x) - Q_1(x)\|_Y \le \Big(\frac{2^{p-1}}{2-2^p} + \frac{2^p}{4-2^p}\Big)\theta\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.3 by taking $\varphi(x,y,z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in X$ with $x \perp y$, and $\alpha = 2^{p-1}$. □

Theorem 2.5. *Let* $\varphi : X^3 \to [0,\infty)$ *be a function such that there exist an* $0 < \alpha < 1$ *with*

$$\varphi(x,y,z) \le \frac{\alpha}{4}\varphi(2x,2y,2z)$$

for all $x, y, z \in X$ *with* $x \perp y$. *Let* $f : X \to Y$ *be a mapping satisfying* (2.2). *Then there exist an orthogonally additive mapping* $L_2 : X \to Y$ *and an orthogonally quadratic mapping* $Q_2 : X \to Y$ *such that*

$$\begin{aligned} &\|f(x) - L_2(x) - Q_2(x)\|_Y \\ &\quad \le \Big(\frac{1}{2(2-\alpha)} + \frac{1}{2(1-\alpha)}\Big)[\varphi(x,0,0) + \varphi(-x,0,0)] \end{aligned} \tag{2.6}$$

for all $x \in X$. *The functions* L_2 *and* Q_2 *are given by*

$$L_2(x) = \lim_{k\to\infty} 2^k f(\frac{x}{2^k}), \quad Q_2(x) = \lim_{k\to\infty} 2^{2k} f(\frac{x}{2^k}),$$

for all $x \in X$.

Proof. It follows from Theorem 2.1 ($\alpha_2 := \frac{\alpha}{2}$) and from Theorem 2.2 ($\alpha_4 := \alpha$) that there exist a unique orthogonally additive mapping $L_2 : X \to Y$, defined by $L_2(x) = \lim_{k\to\infty} 2^k f(\frac{x}{2^k})$, and a unique orthogonally quadratic mapping $Q_2 : X \to Y$, defined by $Q_2(x) = \lim_{k\to\infty} 2^{2k} f(\frac{x}{2^k})$, such that

$$\begin{aligned} \|f_o(x) - L_2(x)\|_Y &\le \frac{1}{(2-\alpha)2}[\varphi(x,0,0) + \varphi(-x,0,0)], \\ \|f_e(x) - Q_2(x)\|_Y &\le \frac{1}{(1-\alpha)2}[\varphi(x,0,0) + \varphi(-x,0,0)] \end{aligned}$$

for all $x \in X$, respectively. Therefore, we obtain the inequality (2.6), which completes the proof. $\square$

Corollary 2.6. [11] *Assume that $(X, \perp)$ is an orthogonality normed space. Let θ be a positive real number and p a real number with $p > 2$. Let $f : X \to Y$ be a mapping satisfying (2.5). Then there exist an orthogonally additive mapping $L_2 : X \to Y$ and an orthogonally quadratic mapping $Q_2 : X \to Y$ such that*

$$\|f(x) - L_2(x) - Q_2(x)\|_Y \leq \Big(\frac{2^{p-1}}{2^p - 2} + \frac{2^p}{2^p - 4}\Big)\theta\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.5 by taking $\varphi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in X$ with $x \perp y$, and $\alpha = 2^{2-p}$. $\square$

Theorem 2.7. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exist $0 < \alpha_1, \alpha_2 < 1$ with*

$$\varphi(x, y, z) \leq 4\alpha_1\varphi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}) \quad \text{and} \quad \varphi(x, y, z) \leq \frac{\alpha_2}{2}\varphi(2x, 2y, 2z)$$

for all $x, y, z \in X$ with $x \perp y$. Let $f : X \to Y$ be a mapping satisfying (2.2). Then there exist an orthogonally additive mapping $L_2 : X \to Y$ and an orthogonally quadratic mapping $Q_1 : X \to Y$ such that

$$\begin{aligned} &\|f(x) - L_2(x) - Q_1(x)\|_Y \\ &\leq \Big(\frac{1}{4(1-\alpha_2)} + \frac{\alpha_1}{2(1-\alpha_1)}\Big)[\varphi(x, 0, 0) + \varphi(-x, 0, 0)] \end{aligned} \tag{2.7}$$

for all $x \in X$. The functions L_2 and Q_1 are given by

$$L_2(x) = \lim_{k\to\infty} 2^k f(\frac{x}{2^k}), \quad Q_1(x) = \lim_{k\to\infty} \frac{1}{2^{2k}} f(2^k x),$$

for all $x \in X$.

Proof. It follows from Theorem 2.1 and from Theorem 2.2 that there exist a unique orthogonally additive mapping $L_2 : X \to Y$ defined by $L_2(x) = \lim_{k\to\infty} 2^k f(\frac{x}{2^k})$ and a unique orthogonally quadratic mapping $Q_1 : X \to Y$ defined by $Q_1(x) = \lim_{k\to\infty} \frac{1}{2^{2k}} f(2^k x)$ such that

$$\begin{aligned} \|f_o(x) - L_2(x)\|_Y &\leq \frac{1}{(1-\alpha_2)4}[\varphi(x, 0, 0) + \varphi(-x, 0, 0)], \\ \|f_e(x) - Q_1(x)\|_Y &\leq \frac{\alpha_1}{(1-\alpha_1)2}[\varphi(x, 0, 0) + \varphi(-x, 0, 0)] \end{aligned}$$

for all $x \in X$, respectively. Therefore, we obtain the inequality (2.7), which completes the proof. $\square$

COROLLARY 2.8. *Assume that $(X,\perp)$ is an orthogonality normed space. Let θ be a positive real number and p a positive real number with $1<p<2$. Let $f: X\to Y$ be a mapping satisfying (2.5). Then there exist an orthogonally additive mapping $L_2: X\to Y$ and an orthogonally quadratic mapping $Q_1: X\to Y$ such that*

$$\|f(x)-L_2(x)-Q_1(x)\|_Y \le \Big(\frac{2^{p-1}}{2^p-2}+\frac{2^p}{4-2^p}\Big)\theta\|x\|^p$$

for all $x\in X$.

Proof. The proof follows from Theorem 2.7 by taking $\varphi(x,y,z)=\theta(\|x\|^p+\|y\|^p+\|z\|^p)$ for all $x,y,z\in X$ with $x\perp y$, and $\alpha_1=2^{p-2}$, $\alpha_2=2^{1-p}$. □

3. Approximate orthogonally additive and orthogonally quadratic mappings in non-Archimedean spaces

Throughout this section, assume that $(X,\perp)$ is a non-Archimedean orthogonality space and that $(Y,\|\cdot\|_Y)$ is a non-Archimedean Banach space. In this section, we introduce the stability results for the equation $Df(x,y,z)=0$ in non-Archimedean spaces with valuation $|2|<1$. Above all, we state the main stability theorems given in the reference [11].

THEOREM 3.1. [11] *Let $\varphi: X^3\to[0,\infty)$ be a function such that there exists an $0<\alpha_1<1$ $(0<\alpha_2<1, resp.)$ with*

$$\begin{aligned}\varphi(x,y,z) &\le |2|\alpha_1\varphi(\frac{x}{2},\frac{y}{2},\frac{z}{2}),\\ \big(\varphi(x,y,z) &\le \frac{\alpha_2}{|2|}\varphi(2x,2y,2z), resp.\big)\end{aligned}$$

for all $x,y,z\in X$ with $x\perp y$. Let $f: X\to Y$ be an odd mapping satisfying

$$\|Df(x,y,z)\|_Y\le\varphi(x,y,z) \tag{3.1}$$

for all $x,y,z\in X$ with $x\perp y$. Then there exists a unique orthogonally additive mapping $L_1: X\to Y$ $(L_2: X\to Y, resp.)$ such that

$$\begin{aligned}\|f(x)-L_1(x)\|_Y &\le \frac{\alpha_1}{|2|-|2|\alpha_1}\varphi(x,0,0),\\ \big(\|f(x)-L_2(x)\|_Y &\le \frac{1}{|2|-|2|\alpha_2}\varphi(x,0,0), resp.\big)\end{aligned}$$

for all $x\in X$.

THEOREM 3.2. [11] *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $0 < \alpha_3 < 1$ ($0 < \alpha_4 < 1, resp.$) with*

$$\begin{aligned} \varphi(x,y,z) &\leq |4|\alpha_3\varphi(\frac{x}{2},\frac{y}{2},\frac{z}{2}), \\ \big(\varphi(x,y,z) &\leq \frac{\alpha_4}{|4|}\varphi(2x,2y,2z), resp.\big) \end{aligned}$$

for all $x,y,z \in X$ with $x \perp y$. Let $f : X \to Y$ be an even mapping satisfying (3.1). *Then there exists a unique orthogonally quadratic mapping $Q_1 : X \to Y$ ($Q_2 : X \to Y, resp.$) such that*

$$\begin{aligned} \|f(x) - Q_1(x)\|_Y &\leq \frac{\alpha_3}{1-\alpha_3}\varphi(x,0,0), \\ \big(\|f(x) - Q_2(x)\|_Y &\leq \frac{1}{1-\alpha_4}\varphi(x,0,0), resp.\big) \end{aligned}$$

for all $x \in X$.

Now, we introduce some additional stability results of orthogonally additive and orthogonally quadratic functional equation $Df(x,y,z) = 0$ in non-Archimedean spaces.

THEOREM 3.3. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $0 < \alpha < 1$ with*

$$\varphi(x,y,z) \leq |4|\alpha\varphi(\frac{x}{2},\frac{y}{2},\frac{z}{2})$$

for all $x,y,z \in X$ with $x \perp y$. Let $f : X \to Y$ be a mapping satisfying (3.1). *Then there exist an orthogonally additive mapping $L_1 : X \to Y$ and an orthogonally quadratic mapping $Q_1 : X \to Y$ such that*

$$\|f(x) - L_1(x) - Q_1(x)\|_Y \leq \frac{\alpha}{|2|(1-\alpha)}\max\{\varphi(x,0,0),\varphi(-x,0,0)\}$$

for all $x \in X$.

Proof. We note that

$$\begin{aligned} \|Df_o(x,y,z)\|_Y &\leq \frac{1}{|2|}\max\{\varphi(x,y,z),\varphi(-x,-y,-z)\}, \\ \|Df_e(x,y,z)\|_Y &\leq \frac{1}{|2|}\max\{\varphi(x,y,z),\varphi(-x,-y,-z)\} \end{aligned} \tag{3.2}$$

for all $x,y,z \in X$ with $x \perp y$. It follows from Theorem 3.1 ($\alpha_1 := |2|\alpha$) and from Theorem 3.2 ($\alpha_3 := \alpha$) that there exist a unique orthogonally additive mapping $L_1 : X \to Y$ and a unique orthogonally quadratic mapping $Q_1 : X \to Y$ such that

$$\begin{aligned}\|f_o(x) - L_1(x)\|_Y &\leq \frac{\alpha}{|2|(1-|2|\alpha)} \max\{\varphi(x,0,0), \varphi(-x,0,0)\},\\ \|f_e(x) - Q_1(x)\|_Y &\leq \frac{\alpha}{|2|(1-\alpha)} \max\{\varphi(x,0,0), \varphi(-x,0,0)\}\end{aligned}$$

for all $x \in X$, respectively. Therefore, we obtain that

$$\begin{aligned}&\|f(x) - L_1(x) - Q_1(x)\|_Y\\ &\leq \max\left\{\frac{\alpha}{|2|(1-|2|\alpha)}, \frac{\alpha}{|2|(1-\alpha)}\right\} \max\{\varphi(x,0,0), \varphi(-x,0,0)\}\\ &= \frac{\alpha}{|2|(1-\alpha)} \max\{\varphi(x,0,0), \varphi(-x,0,0)\}\end{aligned}$$

for all $x \in X$, which completes the proof. □

Corollary 3.4. [11] *Assume that $(X, \perp)$ is a non-Archimedean orthogonality normed space. Let θ be a positive real number and p a positive real number with $p > 2$. Let $f : X \to Y$ be a mapping satisfying (2.5). Then there exist an orthogonally additive mapping $L_1 : X \to Y$ and an orthogonally quadratic mapping $Q_1 : X \to Y$ such that*

$$\|f(x) - L_1(x) - Q_1(x)\|_Y \leq \frac{|2|^{p-1}\theta}{|2|^2 - |2|^p}\|x\|^p$$

for all $x \in X$.

Proof. Taking $\varphi(x,y,z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ and $\alpha = |2|^{p-2}$, we get the desired result by Theorem 3.3. □

Theorem 3.5. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $0 < \alpha < 1$ with*

$$\varphi(x,y,z) \leq \frac{\alpha}{|2|}\varphi(2x,2y,2z)$$

for all $x, y, z \in X$ with $x \perp y$. Let $f : X \to Y$ be a mapping satisfying (3.1). Then there exist an orthogonally additive mapping $L_2 : X \to Y$ and an orthogonally quadratic mapping $Q_2 : X \to Y$ such that

$$\|f(x) - L_2(x) - Q_2(x)\|_Y \leq \frac{1}{|2|^2(1-\alpha)} \max\{\varphi(x,0,0), \varphi(-x,0,0)\}$$

for all $x \in X$.

Proof. It follows from Theorem 3.1 ($\alpha_2 := \alpha$) and from Theorem 3.2 ($\alpha_4 := |2|\alpha$) that there exist a unique orthogonally additive mapping

$L_2 : X \to Y$ and a unique orthogonally quadratic mapping $Q_2 : X \to Y$ such that

$$\begin{aligned}\|f_o(x) - L_2(x)\|_Y &\le \frac{1}{|2|^2(1-\alpha)} \max\{\varphi(x,0,0), \varphi(-x,0,0)\},\\ \|f_e(x) - Q_2(x)\|_Y &\le \frac{1}{|2|(1-|2|\alpha)} \max\{\varphi(x,0,0), \varphi(-x,0,0)\}\end{aligned}$$

for all $x \in X$, respectively. Therefore, we obtain that

$$\begin{aligned}&\|f(x) - L_2(x) - Q_2(x)\|_Y\\ &\le \max\left\{\frac{1}{|2|^2(1-\alpha)}, \frac{1}{|2|(1-|2|\alpha)}\right\} \max\{\varphi(x,0,0), \varphi(-x,0,0)\}\\ &= \frac{1}{|2|^2(1-\alpha)} \max\{\varphi(x,0,0), \varphi(-x,0,0)\}\end{aligned}$$

for all $x \in X$, which completes the proof. □

COROLLARY 3.6. [11] *Assume that $(X, \perp)$ is a non-Archimedean orthogonality normed space. Let θ be a positive real number and p a positive real number with $p < 1$. Let $f : X \to Y$ be a mapping satisfying (2.5). Then there exist an orthogonally additive mapping $L_2 : X \to Y$ and an orthogonally quadratic mapping $Q_2 : X \to Y$ such that*

$$\|f(x) - L_2(x) - Q_2(x)\|_Y \le \frac{|2|^{p-2}\theta}{|2|^p - |2|}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.5 by taking $\varphi(x,y,z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in X$ with $x \perp y$, and $\alpha = |2|^{1-p}$. □

THEOREM 3.7. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exist $0 < \alpha_1, \alpha_2 < 1$ with*

$$\varphi(x,y,z) \le |2|\alpha_1\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \quad \text{and} \quad \varphi(x,y,z) \le \frac{\alpha_2}{|4|}\varphi(2x, 2y, 2z)$$

for all $x, y, z \in X$ with $x \perp y$. Let $f : X \to Y$ be a mapping satisfying (3.1). Then there exist an orthogonally additive mapping $L_1 : X \to Y$ and an orthogonally quadratic mapping $Q_2 : X \to Y$ such that

$$\begin{aligned}&\|f(x) - L_1(x) - Q_2(x)\|_Y\\ &\le \max\left\{\frac{\alpha_1}{|2|^2(1-\alpha_1)}, \frac{1}{|2|(1-\alpha_2)}\right\} \max\{\varphi(x,0,0), \varphi(-x,0,0)\}\end{aligned}$$

for all $x \in X$.

Proof. It follows from Theorem 3.1 and from Theorem 3.2 that there exist a unique orthogonally additive mapping $L_1 : X \to Y$ and a unique orthogonally quadratic mapping $Q_2 : X \to Y$ such that

$$\begin{aligned} \|f_o(x) - L_1(x)\|_Y &\leq \frac{\alpha_1}{|2|^2(1-\alpha_1)} \max\{\varphi(x,0,0), \varphi(-x,0,0)\}, \\ \|f_e(x) - Q_2(x)\|_Y &\leq \frac{1}{|2|(1-\alpha_2)} \max\{\varphi(x,0,0), \varphi(-x,0,0)\} \end{aligned}$$

for all $x \in X$, respectively. Therefore, we obtain that

$$\begin{aligned} &\|f(x) - L_1(x) - Q_2(x)\|_Y \\ &\leq \max\left\{\frac{\alpha_1}{|2|^2(1-\alpha_1)}, \frac{1}{|2|(1-\alpha_2)}\right\} \max\{\varphi(x,0,0), \varphi(-x,0,0)\} \end{aligned}$$

for all $x \in X$, which completes the proof. $\square$

Corollary 3.8. *Assume that $(X, \perp)$ is a non-Archimedean orthogonality normed space. Let θ be a positive real number and p a positive real number with $1 < p < 2$. Let $f : X \to Y$ be a mapping satisfying* (2.5)*. Then there exist an orthogonally additive mapping $L_1 : X \to Y$ and an orthogonally quadratic mapping $Q_2 : X \to Y$ such that*

$$\|f(x) - L_1(x) - Q_2(x)\|_Y \leq \max\left\{\frac{|2|^{p-2}}{|2| - |2|^p}, \frac{|2|^{p-1}}{|2|^p - |2|^2}\right\}\theta\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.7 by taking $\varphi(x,y,z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in X$ with $x \perp y$, and $\alpha_1 = |2|^{p-1}$, $\alpha_2 = |2|^{2-p}$. $\square$

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